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Effect of secondary terms on axisymmetric vibration of circular plates

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SUMMARY

Free transverse vibration of a circular plate is considered by assuming the displacement components as an infinite series in the thickness coordinate. The analysis is done by retaining only the first two terms in each series. The equations of motion are derived by Hamilton's energy principle and the solutions are obtained in terms of Bessel functions. Numerical results are compared with the classical and shear theories which are particular cases of the present theory.

1. Introduction

An up-to-date account of almost all the work performed on vibration of plates is given by Liessa [1,2]. Most of the work is based either on classical theory or on the shear theory given by Mindlin [3]. The present paper considers free transverse axisymmetric vibrations of a circular plate by assuming the displacement components as infinite series in the thickness coordinate and retaining only the first two terms in each component. The shear and classical theories can be considered as particular cases of the present theory. The equations of motion are derived by Hamilton's energy principle. They are solved for harmonic vibrations with the help of auxiliary variables. The solutions are obtained in terms of Bessel functions. The frequency equation is obtained for a plate clamped at the circular edge. Numerical results computed for the first four normal modes of vibration are compared with those of shear and classical theories.

2. Displacements, strains and stresses

We consider a circular plate of radius a, thickness h, density ρ, Young's modulus E and Poisson's ratio ν. The plate is referred to cylindrical coordinates r,θ,z by taking the axis of the plate in the line $r = 0$ and the middle plane of the plate in the plane $z = 0$. The displacement components u,v,w in r,θ,z directions are taken to be

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$$u(r,z,t) = zu_1(r,t) + (z^3/h^2)u_3(r,t) + (z^5/h^4)u_5(r,t) + \ldots,$$

$$v(r,z,t) = 0, \tag{1}$$

$$w(r,z,t) = w_0(r,t) + (z/h)^2 w_2(r,t) + (z/h)^4 w_4(r,t) + \ldots,$$

where t denotes the time.

Retaining only the first two terms in the series of u and w, and using the strain-displacement relations given by Love [4], the non-zero strain components are obtained to be

$$\epsilon_r = zu_{1,r} + (z^3/h^2)u_{3,r}, \quad \epsilon_\theta = zu_1/r + (z^3/h^2)u_3/r$$
$$\epsilon_z = 2zw_2/h^2, \quad \epsilon_{zr} = u_1 + w_{0,r} + (z/h)^2(3u_3 + w_{2,r}), \tag{2}$$

where a comma followed by a suffix denotes the partial differentiation with respect to that variable.

The stress-strain relations are taken to be

$$\sigma_r = (\lambda + 2\mu)\epsilon_r + \lambda(\epsilon_\theta + \epsilon_z), \quad \sigma_\theta = (\lambda + 2\mu)\epsilon_\theta + \lambda(\epsilon_z + \epsilon_r),$$
$$\sigma_z = (\lambda + 2\mu)\epsilon_z + \lambda(\epsilon_r + \epsilon_\theta), \quad \sigma_{zr} = \mu\epsilon_{zr}, \tag{3}$$

where λ and μ are the Lamé constants.

3. Equations of motion

The equations of motion, obtained by applying Hamilton's energy principle in a manner similar to Herrman and Mirsky [5], are

$$m_{r,r} + (m_r - m_\theta)/r - q_r = \rho h^3(u_{1,tt}/12 + u_{3,tt}/80), \tag{4}$$

$$s_{r,r} + (s_r - s_\theta)/r - 3p_r = \rho h^5(u_{1,tt}/80 + u_{3,tt}/448), \tag{5}$$

$$q_{r,r} + q_r/r = \rho h(w_{0,tt} + w_{2,tt}/12), \tag{6}$$

$$p_{r,r} + p_r/r - 2m_z = \rho h^3(w_{0,tt}/12 + w_{2,tt}/80), \tag{7}$$

and the edge conditions are obtained by prescribing one member of each of the following products:

$$m_r u_1, \quad s_r u_3, \quad q_r w_0, \quad p_r w_2, \tag{8}$$

where the stress resultants are given by

$$(m_r, m_\theta, m_z, s_r, s_\theta, q_r, p_r)$$

$$= \int_{-h/2}^{h/2} (z\sigma_r, z\sigma_\theta, z\sigma_z, z^3\sigma_r, z^3\sigma_\theta, \sigma_{zr}, z^2\sigma_{zr}) dz. \tag{9}$$

4. Introduction of auxiliary variables

To get a comparatively simpler system of equations, we introduce auxiliary variables in the following manner:

$$(u_1, u_3, w_0, w_2) = (U_{1,R}, U_{3,R}, aW_0, aW_2) \exp(i\omega t), \tag{10}$$

where $\exp(i\omega t)$ is the time factor for steady-state vibration, ω is the circular frequency, $R = r/a$ and U_1, U_3, W_0 and W_2 are functions of R only.

With the help of eqs. (2), (3), (9) and (10), the eqs. (4) and (5) reduce to equations which can be integrated completely with respect to R to give

$$20[H^2(L\nabla^2 + \Omega^2) - 12K]U_1 + 3[H^2(L\nabla^2 + \Omega^2) - 12K]U_3$$
$$- 240KW_0 + 20MW_2 = 0, \tag{11}$$

$$28[H^2(L\nabla^2 + \Omega^2) - 20K]U_1 + [5H^2(L\nabla^2 + \Omega^2) - 252K]U_3$$
$$- 560KW_0 + 28NW_2 = 0, \tag{12}$$

and the eqs. (6) and (7) reduce to

$$12K\nabla^2 U_1 + 3K\nabla^2 U_3 + 12(K\nabla^2 + \Omega^2)W_0 + (K\nabla^2 + \Omega^2)W_2 = 0, \tag{13}$$

$$20MH^2\nabla^2 U_1 + 3NH^2\nabla^2 U_3 - 20H^2(K\nabla^2 + \Omega^2)W_0$$
$$- [3H^2(K\nabla^2 + \Omega^2) - 80L]W_2 = 0, \tag{14}$$

where

$$K = 1/(2 + 2\nu), \qquad L = (2 - 2\nu)/D, \qquad M = (6\nu - 1)/D,$$
$$N = (10\nu - 3)/D, \qquad D = K/(1 - 2\nu), \qquad H = h/a, \tag{15}$$
$$\nabla^2 \equiv (d^2/dR^2) + (1/R)(d/dR), \qquad \Omega^2 = \rho a^2 \omega^2 / E.$$

In the above equations Ω is the frequency parameter. The arbitrary constants arising from integration in eqs. (11) and (12) are set equal to zero without any loss of generality.

Solving eqs. (11) to (14) for T, where T stands for any one of the variables U_1, U_3, W_0 and W_2, we get

$$(a_0\nabla^8 + a_1\nabla^6 + a_2\nabla^4 + a_3\nabla^2 + a_4)T = 0, \tag{16}$$

where

$$\begin{aligned}
a_0 &= K^2L^2H^6,\\
a_1 &= [2KL(K+L)H^2\Omega^2 - 15K^2L(6K+5M-7N) + KL(93.75M^2\\
&\quad + 78.75N^2 - 60L^2 - 157.5MN)]H^4,\\
a_2 &= [(K^2+L^2+4KL)H^4\Omega^4 + 15\{K^2(6K+18L-5M-10.5N)\\
&\quad - KL(8L-7N+5M) + K(5.25N^2 - 10.5MN + 6.25M^2)\\
&\quad + L(6.25M^2 - 4L^2 - 10.5ML + 5.25N^2)\}H^2\Omega^2\\
&\quad + 2100K^2(4L^2 - K^2 - 2KM - M^2)]H^2,\\
a_3 &= 2(K+L)H^6\Omega^6 + (78.75N^2 - 270K^2 - 120L^2 - 240KL\\
&\quad + 105KN - 75KM - 157.5MN + 93.75M^2)H^4\Omega^4\\
&\quad + \{2100K^2(2L-K-M) + 45K(240L^2 - 21N^2 + 70MN - 105M^2\}H^2\Omega^2,\\
a_4 &= H^6\Omega^8 - 60(3K+L)\Omega^6H^4 + 240K(7K+45L)\Omega^4H^2 - 100800K^2L^2\Omega^2.
\end{aligned} \tag{17}$$

5. Solution

The general solution of the Bessel equation

$$(\nabla^2 + p^2)T = 0. \tag{18}$$

will also be a solution of eq. (16), provided

$$a_0p^8 - a_1p^6 + a_2p^4 - a_3p^2 + a_4 = 0. \tag{19}$$

This equation is a bi-quadratic equation in p^2. In actual computation, out of the four roots of this equation, one comes out positive and the other three come out negative. If these roots be denoted by p_1^2, $-p_2^2$, $-p_3^2$ and $-p_4^2$, then the general solutions for U_1, U_3, W_0 and W_2 can be taken as

$$(U_1, U_3, W_0, W_2) = \sum_{\alpha=1}^{4} (1, B_\alpha, C_\alpha, D_\alpha) A_\alpha F_\alpha(Rp_\alpha), \tag{20}$$

where $F_1 = J_0$, $F_\alpha = I_0$ for $\alpha = 2, 3, 4$. J_0 and I_0 denote the Bessel and modified Bessel functions of first kind and order zero. The A_α's are arbitrary constants; B_α, C_α and D_α can be obtained by solving any three of the eqs. (11) to (14) after substituting the solutions (20) in them.

6. Frequency equation

If we take the plate clamped at the circular edge then the edge conditions are

$$u_1 = u_3 = w_0 = w_2 = 0, \quad \text{at } R = 1 \tag{21}$$

Substituting solutions (20) in (10) and then using (21), we get

$$\begin{aligned}
&J_1(p_1)A_1 - I_1(p_2)A_2 - I_1(p_3)A_3 - I_1(p_4)A_4 = 0,\\
&B_1J_1(p_1)A_1 - B_2I_1(p_2)A_2 - B_3I_1(p_3)A_3 - B_4I_1(p_4)A_4 = 0,\\
&C_1J_0(p_1)A_1 + C_2I_0(p_2)A_2 + C_3I_0(p_3)A_3 + C_4I_0(p_4)A_4 = 0,\\
&D_1J_0(p_1)A_1 + D_2I_0(p_2)A_2 + D_3I_0(p_3)A_3 + D_4I_0(p_4)A_4 = 0.
\end{aligned} \tag{22}$$

Eliminating A_1 to A_4 from these equations, we get the frequency equation in the form of a fourth-order determinant equated to zero. This determinant involves the frequency parameter implicitly. Successive zeros of the determinant give frequencies of vibration for successive normal modes.

Retaining only the first term in the series for u and w given in eqs. (1), proceeding in a manner similar to above and introducing a shear constant in the shear force, we can get the frequency equation for the shear theory. The frequency equation for the classical theory can be obtained by replacing u_1 by $-w_{0,r}$ and retaining only the first term in u and w.

7. Numerical results

The frequency parameter Ω versus H, computed for the present theory (P-theory), the shear theory (S-theory) and the classical theory (C-theory), for the first four normal modes of vibration is plotted in Fig. 1. It is clear that the values of Ω in the S-theory are smaller than the values in the C-theory but greater than those of P-theory in all the four modes of vibration and for all values of H up to $H = 2.4$. The difference between the S- and C-theories goes on increasingly rapidly as we go to higher values of H or to higher modes of vibration. But the

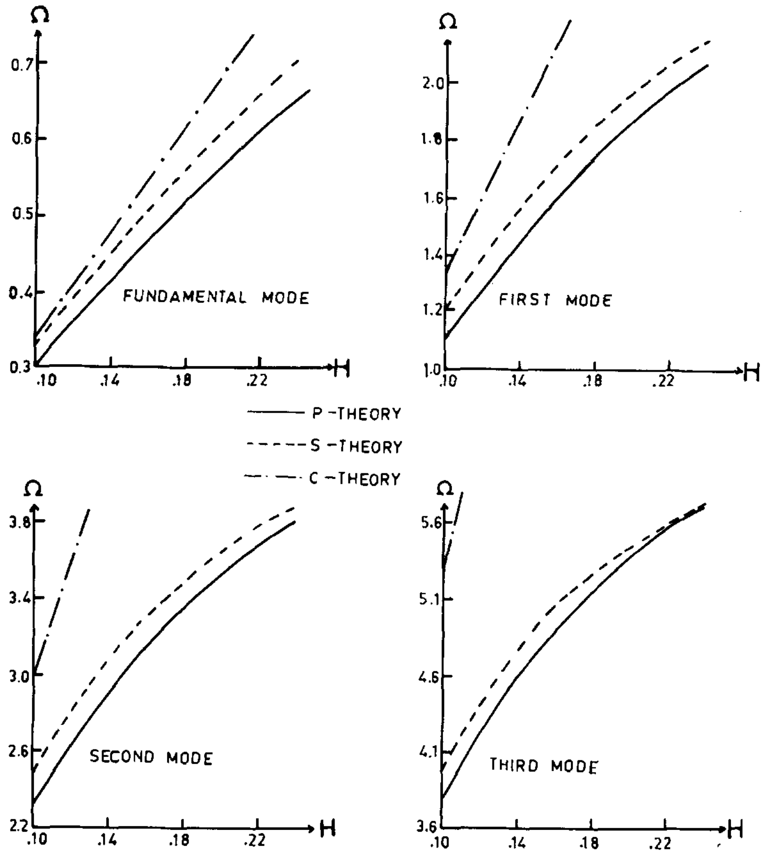

Fig. 1. Ω versus H.

difference between S and P-theories in the fundamental mode increases a little with an increase in H. It remains almost constant in the first mode and it decreases in the second and third mode.

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